

A Variation on the Homological Nerve Theorem

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Abstract

An equivalent but useful version of the Homological Nerve Theorem is proved.

1 Introduction

Let X be a polyhedron, $F = \{A_1, \dots, A_m\}$ a polyhedral cover of X and let $N = N(F)$ be the nerve of the family F . Denote by $N^{(k)}$ the k -skeleton of the simplicial complex N . During these paper we are going to use reduced homology with coefficients in a field. We say that $A \subset X$ is ρ -acyclic if $\tilde{H}_*(A) = 0$ for $* \leq \rho$. Furthermore, $\tilde{H}_{-1}(A) = 0$ means that A is not empty.

The Homological Nerve Theorem, as stated by Meshulam in [3], claims the following:
Homological Nerve Theorem. Suppose that for every $\sigma \in N^{(k)}$,

- \bigcap_{σ} is $(k - |\sigma| + 1)$ -acyclic. Then,

$$\text{Rank } \tilde{H}_{k+1}(N) \leq \text{Rank } \tilde{H}_{k+1}(X).$$

and for every $0 \leq j \leq k$,

$$\tilde{H}_j(N) = \tilde{H}_j(X).$$

The purpose of this paper is to prove the following equivalent but useful variation of the Homological Nerve Theorem:

Theorem 1. Suppose that for every $\sigma \in N^{(k)}$,

- $\tilde{H}_{k-|\sigma|+1}(\bigcap_{\sigma}) = 0$. Then,
1. $\text{Rank } \tilde{H}_{k+1}(N) \leq \text{Rank } \tilde{H}_{k+1}(X)$,
 2. $\text{Rank } \tilde{H}_k(X) \leq \text{Rank } \tilde{H}_k(N)$.

In many applications of the Homological Nerve Theorem the useful conclusion is that $\tilde{H}_{k+1}(X) = 0$ implies $\tilde{H}_{k+1}(N) = 0$. So Theorem 1 helps to improve these results since the hypothesis needed to achieve this conclusion are much weaker. In Section 3, we give a couple of examples of this fact.

2 Killing the homology groups

Let X be a polyhedron and let $f : S^d \rightarrow X$ be a PL-map. Denote by $X \cup_f B^{d+1}$ the polyhedron obtained by attaching the $(d+1)$ -cell to X . Then, if $u \in \tilde{H}_d(S^d)$ is the fundamental class, and $f_*(u) \in \tilde{H}_d(X)$ is not zero, we have that for every $* \neq d$,

$$\begin{aligned}\tilde{H}_*(X) &= \tilde{H}_*(X \cup_f B^{d+1}), \text{ and} \\ \tilde{H}_d(X \cup_f B^{d+1}) &= \frac{\tilde{H}_d(X)}{\langle f_*(u) \rangle}.\end{aligned}$$

The next lemma is essential in several proofs of this paper. Its proof is part of the "folklore" and we include it by completeness.

Lemma 1.. Let A be a polyhedron. Then by attaching λ -cells to A , $\lambda \leq k$, we obtain a polyhedron \tilde{A} containing A such that

- \tilde{A} is $(k-1)$ -acyclic, and
- $\tilde{H}_i(A) = \tilde{H}_i(\tilde{A})$, for $i \geq k$.

Proof. If $k = 1$, the theorem is trivial because we just have to connect components with arcs. If $k = 2$, Let $g_1, \dots, g_t : S^1 \rightarrow A$ continuous maps such that $(g_i)_*(u) \neq 0$ and $\{(g_i)_*(u)\}_1^t$ generates $\tilde{H}_1(A)$, where $u \in \tilde{H}_1(S^1)$ is the fundamental class. Then, by attaching 2-cells to A , via $g_1, \dots, g_t : S^1 \rightarrow A$ we achieve our purpose. Suppose now $k = 3$. We first kill the fundamental group. Let $\psi_1, \dots, \psi_t : (S^1, 1) \rightarrow (A, *)$ be continuous maps such that $\{(\psi_i)_*(u)\}_1^t$ generates $\pi_1(A, *)$, where now $u \in \pi_1(S^1, *)$ is the generator. Then, by attaching 2-cells to A , via $\psi_1, \dots, \psi_t : S^1 \rightarrow A$ we obtain \tilde{A} , killing the fundamental group of A but perhaps creating 2-dimensional homology. In other words $\pi_1(\tilde{A}, *) = 0 = \tilde{H}_1(\tilde{A})$, and $\tilde{H}_*(A) = \tilde{H}_*(\tilde{A})$, for $* \geq 3$. Now, by the Hurewicz Theorem $\pi_2(\tilde{A}, *) = \tilde{H}_2(\tilde{A}; Z)$. So let $\psi_1, \dots, \psi_t : (S^2, 1) \rightarrow (\tilde{A}, *)$ be continuous maps such that $\{(\psi_i)_*(u)\}_1^t$ generates $\pi_2(\tilde{A}, *)$ and $(\psi_i)_*(u) \neq 0$, where this time $u \in \pi_2(S^2, *)$ is the generator. Then, by attaching 2-cells to \tilde{A} , via $\psi_1, \dots, \psi_t : S^2 \rightarrow \tilde{A}$ we obtain $\tilde{\tilde{A}}$. First of all, note that $\tilde{H}_2(\tilde{A}, Z) = 0$ but also that, again by the Hurewicz Theorem, $(\psi_i)_*(u) \neq 0 \in \tilde{H}_2(\tilde{A}; Z)$. Therefore, by the universal coefficient Theorem for homology, since $\tilde{H}_1(\tilde{A}) = 0$, we have that $\tilde{H}_2(\tilde{\tilde{A}}) = 0$. Furthermore, again by the universal coefficient Theorem, $(\psi_i)_*(u) \neq 0 \in \tilde{H}_2(\tilde{\tilde{A}})$ and consequently $\tilde{H}_*(\tilde{\tilde{A}}) = \tilde{H}_*(A)$, for $* \geq 3$. The proof of the theorem for higher k 's is completely analogous to the case $k = 3$. \square

Proof of Theorem 1.

The idea is to carefully kill the homology of the A_i 's and its intersections by attaching cells in such a way that we can use the Homological Nerve Theorem. We will do it in such a way that we do not modify the nerve of the family and in such a way we do not change the $(k+1)$ -dimensional homology of the resulting new X .

Inductive Claim C_r . $1 \leq r \leq k+1$.

It is possible to construct polyhedra \tilde{A}_i , and define $\tilde{F} = \{\tilde{A}_i\}_1^m$ and $\tilde{X} = \bigcup_1^m \tilde{A}_i$, in such a way that:

1. $A_i \subset \tilde{A}_i$,
2. $N = N(\tilde{F})$,
3. $\tilde{H}_{k+1}(\tilde{X}) = \tilde{H}_{k+1}(X)$,
4. $\text{Rank } \tilde{H}_k(X) \leq \text{Rank } \tilde{H}_k(\tilde{X})$,
5. for every $\sigma \in N^{(k)}$, $\tilde{H}_{k-|\sigma|+1}(\bigcap_{i \in \sigma} \tilde{A}_i) = 0$,
6. if $\sigma \in N^{(k)}$ and $|\sigma| \geq r$, $\bigcap_{i \in \sigma} \tilde{A}_i$ is $(k - |\sigma| + 1)$ -acyclic.

By defining $A_i = \tilde{A}_i$, C_{k+1} is clearly true. Next we shall prove that if C_{r+1} is true then C_r is also true. Note that if C_1 is true, then we obtain our result by applying the Homological Nerve Theorem, because on one side $\tilde{H}_{k+1}(\tilde{X}) = \tilde{H}_{k+1}(X)$, $\tilde{H}_{k+1}(N) = \tilde{H}_{k+1}(N(\tilde{F}))$ and $\text{Rank } \tilde{H}_{k+1}(N(\tilde{F})) \leq \text{Rank } \tilde{H}_{k+1}(\tilde{X})$ and in the other side, $\text{Rank } H_k(X) \leq \text{Rank } \tilde{H}_k(\tilde{X}) = \text{Rank } \tilde{H}_k(N)$,

Suppose the inductive claim C_{r+1} is true. Let us first fix $\sigma \in N^{(k)}$ with $|\sigma| = r$. Hence, by (4), $\tilde{H}_{k-r+1}(\bigcap_{i \in \sigma} \tilde{A}_i) = 0$. By Lemma 1, we can kill of the $(k - r)$ -homology of $\bigcap_{i \in \sigma} \tilde{A}_i$, by attaching λ -cells of dimension smaller or equal than $k - r + 1$. So, for $i \in \sigma$, we obtain $\tilde{\tilde{A}}_i$ from \tilde{A}_i , by attaching the same λ -cells of dimension smaller or equal than $k - r + 1$, in such a way that $\bigcap_{i \in \sigma} \tilde{\tilde{A}}_i$ is $(k - r + 1)$ -acyclic. Finally, for $\rho \in \{1, \dots, m\} - \sigma$, let $\tilde{\tilde{A}}_\rho = \tilde{A}_\rho$ and let $\tilde{\tilde{F}} = \{\tilde{\tilde{A}}_i\}_1^m$ and $\tilde{\tilde{X}} = \bigcup_1^m \tilde{\tilde{A}}_i$.

Our next purpose is to prove that for $S \subset \{1, \dots, m\}$,

$$\bigcap_{i \in S} \tilde{A}_i = \bigcap_{i \in S} \tilde{\tilde{A}}_i.$$

whenever S is not contained in σ , and $\bigcap_{i \in S} \tilde{\tilde{A}}_i$ is obtained from $\bigcap_{i \in S} \tilde{A}_i$ by attaching λ -cells of dimension smaller or equal than $k - r + 1$, whenever $S \subset \sigma$.

Suppose first S is not contained in σ , then $\bigcap_{i \in S} \tilde{A}_i = \bigcap_{i \in S} \tilde{\tilde{A}}_i$, because if $j \in S - \sigma$, then $\tilde{A}_j \cap \tilde{X} = \tilde{\tilde{A}}_j$. Then $\bigcap_{i \in S} \tilde{A}_i = (\bigcap_{i \in S - \{j\}} \tilde{A}_i) \cap \tilde{A}_j = (\bigcap_{i \in S - \{j\}} \tilde{A}_i) \cap (\tilde{\tilde{A}}_j \cap \tilde{X}) = (\bigcap_{i \in S} \tilde{\tilde{A}}_i) \cap \tilde{X} = \bigcap_{i \in S} (\tilde{\tilde{A}}_i \cap \tilde{X}) = \bigcap_{i \in S} \tilde{\tilde{A}}_i$.

On the other hand, if $S \subset \sigma$, then by definition of the $\tilde{\tilde{A}}_i$'s, we have that $\bigcap_{i \in S} \tilde{\tilde{A}}_i$ is obtained from $\bigcap_{i \in S} \tilde{A}_i$ by attaching λ -cells of dimension smaller or equal than $k - r + 1$.

So, here are some important consequences of the above:

- $N(\tilde{\tilde{F}}) = N$.
- $\tilde{\tilde{H}}_{k+1}(\tilde{\tilde{X}}) = \tilde{\tilde{H}}_{k+1}(X)$.
- $\text{Rank } \tilde{\tilde{H}}_k(X) \leq \text{Rank } \tilde{\tilde{H}}_k(\tilde{\tilde{X}})$.
- For every $\tau \in N^{(k)}$ with $|\tau| \geq r + 1$, we have that $\bigcap_{i \in \tau} \tilde{\tilde{A}}_i = \bigcap_{i \in \tau} \tilde{A}_i$.

- For every $\tau \in N^{(k)}$ with $|\tau| = r$ and $\tau \neq \sigma$, we have that $\bigcap_{i \in \tau} \tilde{\tilde{A}}_i = \bigcap_{i \in \tau} \tilde{A}_i$.
- For every $\tau \in N^{(k)}$ with $|\tau| < r$, we have that $\tilde{H}_{k-|\tau|+1}(\bigcap_{i \in \tau} \tilde{\tilde{A}}_i) = 0$. This is so, because either $\bigcap_{i \in \tau} \tilde{\tilde{A}}_i = \bigcap_{i \in \tau} \tilde{A}_i$ or $\bigcap_{i \in \tau} \tilde{\tilde{A}}_i$ is obtained from $\bigcap_{i \in \tau} \tilde{A}_i$ by attaching λ -cells of dimension $k - r + 1 < k - |\tau| + 1$.

By performing one by one this construction, to every $\sigma \in N^{(k)}$ with $|\sigma| = r$, we obtain that C_r is true. This completes the proof of the theorem. \square

Note that if for every $\sigma \in N^{(k)}$,

- \bigcap_{σ} is $(k - |\sigma| + 1)$ -acyclic.

Then, for every $0 \leq k' \leq k$, the following is true:
for every $\tau \in N^{(k')}$, $\tilde{H}_{k'-|\tau|+1}(\bigcap_{\sigma} \tilde{\tilde{A}}_i) = 0$. This implies, by Theorem 1, that for every $0 \leq k' \leq k$,

1. $\text{Rank } \tilde{H}_{k'+1}(N) \leq \text{Rank } \tilde{H}_{k'+1}(X)$,
2. $\text{Rank } \tilde{H}_{k'}(X) \leq \text{Rank } \tilde{H}_{k'}(N)$,

and therefore that for every $0 \leq k' \leq k$,

$$\begin{aligned} \tilde{H}_{k'}(N) &= \tilde{H}_{k'}(X) \quad \text{and} \\ \text{Rank } \tilde{H}_{k+1}(N) &\leq \text{Rank } \tilde{H}_{k+1}(X), \end{aligned}$$

thus proving that Theorem 1 is equivalent to the Homological Nerve Theorem.

3 Some consequences of Theorem 1

In many applications of the Homological Nerve Theorem the useful conclusion is that $\tilde{H}_{k+1}(X) = 0$ implies $\tilde{H}_{k+1}(N) = 0$. The purpose of this section is to show a couple of examples in which the use of Theorem 1 instead of the Homological Nerve Theorem helps to improve these results since the hypothesis needed to achieve the conclusions are much weaker.

Let us start proving that the Topological Helly Theorem, obtained by Kallai and Meshulam in [2], can be derived from the Homological Nerve Theorem.

Topological Helly Theorem Let $F = \{A_1, \dots, A_m\}$ be a collection of polyhedra in R^d , $m \geq d + 2$. Suppose that for every subfamily F' of F of size n , $1 \leq n \leq d + 1$,

$$\bigcap_{F'} \text{ is } (d - n)\text{-acyclic,}$$

then

$$\bigcap_F \neq \emptyset.$$

Proof. As usual for a proof of a Helly type theorem, the proof follows by induction, and in this case also by a simple Mayer Vietories argument, from the case $m = d + 2$.

Suppose $m = d + 2$ and suppose also $\bigcap_F = \emptyset$. Then $N(F)$ is the boundary of a simplex with $d + 2$ vertices and hence homeomorphic to the d -sphere. On the other hand, if $k = d - 1$, for every $\sigma \in N^{(d-1)}$, \bigcap_σ is $(d - |\sigma|)$ -acyclic. Then, by the first conclusion of the Homological Nerve theorem, $\tilde{H}_d(N(F)) = 0$ because $\tilde{H}_d(\bigcup_F) = 0$, but this is a contradiction to the fact that $N(F)$ is a d -sphere. \square

As we can see, we only used the first conclusion of the Homological Nerve Theorem, so exactly the same proof but now using Theorem 1 instead of the Homological Nerve Theorem yields the following topological Helly-type Theorem, first proved in [5].

Theorem 2 Let $F = \{A_1, \dots, A_m\}$ be a collection of polyhedra in R^d , $m \geq d + 2$. Suppose that for every subfamily F' of F of size n , $1 \leq n \leq d + 1$,

$$\tilde{H}_{d-n}(\bigcap_{F'}) = 0,$$

then

$$\bigcap_F \neq \emptyset.$$

Let K be a simplicial complex. Suppose the vertices of K are painted with $I = \{1, \dots, m\}$ colours, that is, there is a partition of the set of vertices of K ; $V(K) = V_1 \sqcup \dots \sqcup V_m$. A simplex $\sigma = \{v_1, \dots, v_m\} \subset V(K)$ is rainbow if it contains exactly one vertex of every colour. Finally, let $S \subset I$ be a subset of colours. Let $V_S \subset V(K)$ be the set of vertices of K painted with a colour in S and let K_S be the subcomplex of K generated by vertices of V_S .

The following theorem was proved by Meshulam [4] and also Aharoni-Berger [1].

Theorem 3. Let K be a simplicial complex and suppose the vertices of K are painted with $I = \{1, \dots, m\}$ colours. Then K contains a rainbow simplex provided

$$K_S \text{ is } (s - 2)\text{-acyclic,}$$

for every subset $S \subset I$ of s colours, $1 \leq s \leq m$.

In the proof of Theorem 3, given Meshulam [4], he used the first conclusion of the Homological Nerve Theorem to conclude the existence of a rainbow simplex, so exactly the same proof but now using Theorem 1 instead of the Homological Nerve Theorem yield the following improvement of Theorem 3. For more about this kind of Sperner-type Theorems, see [6].

Theorem 4. Let K be a simplicial complex and suppose the vertices of K are painted with $I = \{1, \dots, m\}$ colours. Then K contains a rainbow simplex provided

$$\tilde{H}_{s-2}(K_S) = 0,$$

for every subset $S \subset I$ of s colours, $1 \leq s \leq m$.

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